



MATHEMATICS SPECIALIST : UNITS 1 & 2, 2023

Tuesday 14th March (T1W7)

Test 1 – Proofs and Counting Techniques (10%)

2.3.1, 1.1.1, 1.1.3 – 1.1.5, 1.2.1 – 1.2.9

Time Allowed 30 minutes	First Name <i>Marking Guide</i>	Surname	Marks / 29 marks
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Circle your Teacher's Name:

Ms Chua

Mr Koulianos

Mr Liu

Ms Robinson

Assessment Conditions: (N.B. Sufficient working out must be shown to gain full marks)

- ❖ Calculators: Not Allowed
- ❖ Formula Sheet: Provided
- ❖ Notes: Not Allowed

PART A – CALCULATOR FREE

Question 1 (6 marks)

Consider the conjecture below:

$$\text{If } y^4 < 2xy^2 - x^2, \text{ then } x \notin \mathbb{R} \text{ or } y \notin \mathbb{R}.$$

a) Write the converse, inverse and contrapositive of this conjecture in the space provided.

(3 marks)

Converse	If $x \notin \mathbb{R}$ or $y \notin \mathbb{R}$, then $y^4 < 2xy^2 - x^2$. ✓ R/W
Inverse	If $y^4 \geq 2xy^2 - x^2$, then $x \in \mathbb{R}$ and $y \in \mathbb{R}$. ✓ R/W
Contrapositive	If $x \in \mathbb{R}$ and $y \in \mathbb{R}$, then $y^4 \geq 2xy^2 - x^2$. ✓ R/W

b) Prove that the **contrapositive statement** is true.

(3 marks)

$$\text{RTP: } y^4 \geq 2xy^2 - x^2, \text{ i.e. } y^4 - 2xy^2 + x^2 \geq 0.$$

$$\text{LHS} = y^4 - 2xy^2 + x^2$$

$$= (y^2)^2 - 2xy^2 + x^2$$

$$= (y^2 - x)^2$$

$$\geq 0 \text{ since } x \in \mathbb{R} \text{ and } y \in \mathbb{R}.$$

$$\text{Hence } y^4 - 2xy^2 + x^2 \geq 0 \text{ as req.}$$

$$\text{i.e. if } x \in \mathbb{R} \text{ and } y \in \mathbb{R}, \text{ then } y^4 \geq 2xy^2 - x^2$$

$\therefore$ contrapositive is true.

- ✓ rearranges inequality to be proved
- ✓ expresses as a perfect square
- ✓ indicates this is ≥ 0 (no need to specifically refer to real numbers)

Allow equivalent marks if students choose to start with the perfect square (being greater or equal to zero) and rearrange

Question 2 (5 marks)

Prove the following conjecture:

For $x \in \mathbb{Z}$, $x^2 - 11x + 28$ is never prime.

$$x^2 - 11x + 28 = (x-4)(x-7)$$

For this to be prime,

$$\begin{array}{ll} \text{either } x-4=1 & \text{or } x-7=1 \\ \Rightarrow x=5 & \Rightarrow x=8 \end{array}$$

- ✓ factorises the quadratic
- ✓ indicates the factors need to be 1 for prime
- ✓ solves both possible x-values
- ✓ substitutes both values and shows a non-prime
- ✓ final conclusion

$$\begin{aligned} \text{But, if } x=5, \quad x^2 - 11x + 28 &= 1 \times (5-7) \\ &= -2 < 0 \quad \therefore \text{not prime} \end{aligned}$$

$$\begin{aligned} \text{if } x=8, \quad x^2 - 11x + 28 &= (8-4) \times 1 \\ &= 4 \\ &= 2 \times 2 \quad \therefore \text{not prime.} \end{aligned}$$

Hence, for $x \in \mathbb{Z}$, $x^2 - 11x + 28$ is never prime.**Question 3 (3 marks)**

A set of cards is numbered with all the multiples of 6 between 200 and 900 inclusive. Determine the minimum number of cards which must be selected to ensure that at least 8 cards in the selection have the same last digit. Justify your answer.

5 pigeonholes: last digits for multiplier of 6 can only be
0, 2, 4, 6 or 8

worst case scenario:

$$\text{select 7 of each} = 7 \times 5 = 35 \text{ cards. (pigeons)}$$

By Pigeonhole principle, the next card (pigeon) selected will mean that there are at least 8 of at least one of these last digits.

Hence, minimum # to select is 36 cards.

- ✓ identifies pigeonholes
- ✓ uses worst case scenario
- ✓ adds 1 to get the final answer of 36

Question 4 (6 marks)

- a) How many subsets, excluding the empty set and the original set itself, can be made from the set $\{6, 7, 8, 9, W, X, Y, Z\}$?

8 elements $\Rightarrow$ #subsets $= 2^8 - 2$

✓ expresses as $2^8 - 2$ (or sum of combinations)
✓ evaluates correctly

(2 marks)

$= 256 - 2 = \boxed{254 \text{ subsets.}}$

- b) Evaluate ${}^{50}P_3 \div {}^7P_2$.

(2 marks)

$= \frac{50!}{47!} \div \frac{7!}{5!}$

$= \frac{50 \times 49 \times 48}{7 \times 6}$

✓ expresses using the correct products

✓ evaluates correctly

No mark just for substituting into the permutation formula

$= \boxed{2800}$

- c) An excerpt of Pascal's triangle has been given, below right. Use this to evaluate $\binom{27}{3} - \binom{29}{27}$.

(2 marks)

$\binom{27}{3} - \binom{29}{27}$

$= \binom{27}{3} - \binom{29}{2}$

$= 2925 - 406$

$= \boxed{2519}$

✓ identifies correct values

✓ evaluates correctly

max 1 mark for answer only

	1	25	300	
$\binom{27}{0}$	1	$\binom{27}{1}$	26	$\binom{27}{2}$
1	27	351	325	$\binom{27}{3}$
				2925
$\binom{29}{1}$	28	$\binom{29}{2}$	378	3276
29	406	3654	23751	
	435	4060	27405	

Question 5 (4 marks)

For each of the following statements, state whether it is true or false, and give an example or counterexample to justify your conclusion. You do NOT need to complete any proof.

- a) If an animal is a bird, then it can fly.

(2 marks)

False.

A kiwi is a bird, but it cannot fly.

✓ identifies false

✓ gives an appropriate counterexample

- b) All linear graphs have exactly two axes intercepts.

(2 marks)

False.

$y = 4x$ is a linear graph, but only has one axis intercept at $(0,0)$.

✓ identifies false

✓ gives an appropriate counterexample

Question 6 (5 marks)

Solve for n if $12 \binom{n}{2} = \binom{2n}{3}$. A trial and error method will not receive any marks.

$$12 \times \frac{n!}{2! (n-2)!} = \frac{(2n)!}{3! (2n-3)!}$$

$$\cancel{6} \cancel{12} \times \cancel{n} \times (n-1) = \frac{\cancel{2n} \times (2n-1) \times (2n-2)}{\cancel{6}_3}$$

$$\cancel{6} \cancel{n} (n-1) = \frac{\cancel{n} (2n-1)(2n-2)}{3}$$

$$6n-6 = \frac{4n^2-6n+2}{3}$$

$$18n-18 = 4n^2-6n+2$$

$$4n^2-24n+20=0$$

$$n^2-6n+5=0$$

$$(n-1)(n-5)=0$$

$$\therefore n=1, 5$$

But since eqn involves $\binom{n}{2}$ & $\binom{2n}{3}$,

require $n \geq 2$ & $2n \geq 3$

$\Rightarrow n=1$ is invalid.

$$\therefore \boxed{n=5}$$

✓ expresses using the combination formula

deduct a mark if only $2n!$ not $(2n)!$

✓ simplifies the factorials to achieve polynomial

✓ simplifies to the quadratic equation

✓ solves both values of n

✓ indicates only the $n=5$ solution is valid

(no explanation needed)

An answer without sufficient algebraic reasoning is worth 0 marks.



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Test 1 – Proofs and Counting Techniques (10%)

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Time Allowed 20 minutes	First Name <i>Marking Guide</i>	Surname	Marks / 21 marks
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Circle your Teacher's Name:

Ms Chua

Mr Koulianos

Mr Liu

Ms Robinson

Assessment Conditions: (N.B. Sufficient working out must be shown to gain full marks)

- ❖ Calculators: Allowed
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PART B – CALCULATOR ASSUMED

Question 7 (4 marks)

a) What is the negation of the statement: $\forall x \in \mathbb{Z}, x^2 - 3x + 5$ is odd.

(2 marks)

$\exists x \in \mathbb{Z}$ such that $x^2 - 3x + 5$ is even.

- ✓ changes to 'there exists'
- ✓ negates the remainder of the statement

b) Prove the following statement is true.

$\exists a \in \mathbb{R}$ such that $\sqrt{a^3 - 4a + 1} \in \mathbb{Q}$

other examples.

(2 marks)

$$\text{If } a=2, \sqrt{a^3 - 4a + 1} = \sqrt{8 - 8 + 1} = \sqrt{1}$$

a	3	4	10	12
$\sqrt{a^3 - 4a + 1}$	4	7	31	41

$$= \sqrt{1}$$

$$= 1 \in \mathbb{Q}, \text{ i.e. rational}$$

- ✓ uses an appropriate value for a
- ✓ evaluates and indicates it is a rational # result

Hence, since an a-value can be found, statement is true.

Question 8 (3 marks)

The local high schools want to form an interschool volleyball team of 6 by randomly selecting players from their top teams. The table below shows the number of players available in each school.

At least one student must be selected from each school, and there must be an equal number selected from Rossmoynce and Willetton. How many teams are possible?

School	Number of Players Available
Rossmoynce	12
Willetton	8
Applecross	5

Options: 1R, 1W, 4A or 2R, 2W, 2A.

$$\Rightarrow \binom{12}{1} \times \binom{8}{1} \times \binom{5}{4} + \binom{12}{2} \times \binom{8}{2} \times \binom{5}{2}$$

$$= 480 + 18480$$

$$= \boxed{18960 \text{ teams}}$$

- ✓ indicates correct product of combinations for one option
- ✓ indicates correct product of combinations for the other option
- ✓ correct use of addition principle and evaluates

Question 9 (4 marks)

Four of the letters from the word CATCHCRY are ordered. How many arrangements are possible?

case (i) 4 diff. letters

$$\binom{6}{4} \times 4! = {}^6P_4 = 360$$

C A T H R Y
C
C

case (ii) 2 C's & 2 diff.

$$\binom{5}{2} \times \frac{4!}{2!} = 120$$

case (iii) 3 C's & 1 diff.

$$\binom{5}{1} \times \frac{4!}{3!} = 20$$

✓ correct number of arrangements for first case

✓ correct number for second case

✓ correct number for third case

✓ total

F/T for total only if cases are appropriate

$$\therefore \text{Total \# arrangements} = 360 + 120 + 20 \\ = \boxed{500}$$

Question 10 (4 marks)

Consider the set of integers between 1 and 500 inclusive.

How many of these integers are a multiple of 4, a multiple of 6 or a multiple of 7?

$$\text{mult. of 4: } \left\lfloor \frac{500}{4} \right\rfloor = 125$$

$$6: \left\lfloor \frac{500}{6} \right\rfloor = 83$$

$$7: \left\lfloor \frac{500}{7} \right\rfloor = 71$$

$$\text{mult. of 4 \& 6: } \left\lfloor \frac{500}{12} \right\rfloor = 41$$

$$4 \& 7: \left\lfloor \frac{500}{28} \right\rfloor = 17$$

$$6 \& 7: \left\lfloor \frac{500}{42} \right\rfloor = 11$$

$$\text{mult. of 4, 6 \& 7: } \left\lfloor \frac{500}{84} \right\rfloor = 5$$

$$\text{Total} = 125 + 83 + 71 - 41 - 17 - 11 + 5 \\ = \boxed{215}$$

✓ calculates the number of 'singles' correctly

✓ calculates the number of 'pairs' correctly

✓ calculates the number of 'triples' correctly

✓ applies inclusion-exclusion for correct total

F/T for total only if cases are appropriate

Question 11 (6 marks)

A group of trees are to be planted in a new local park: 3 eucalyptus, 2 golden wattle and 5 paperbark. Assume that all of the trees of the same type are indistinguishable from each other, e.g. all 3 of the eucalyptus trees can be considered the same.

- a) If the trees are to be planted in a single row, how many arrangements involve the golden wattle trees at the very ends of the row (i.e. one at each end)?

(1 mark)

$$1 \times \frac{8!}{3! \times 5!} \times 1 = \boxed{56} \text{ arrangements} \quad \checkmark \text{ R/W}$$

- b) If the trees are to be planted in a single row, how many arrangements are there where the three eucalyptus trees are **not** all kept together?

(2 marks)

$$\left. \begin{array}{l} \text{total \# arrangements (no restrictions)} \\ \frac{10!}{2! \cdot 3! \cdot 5!} = 2520 \\ \\ \text{total \# with 3 adjacent eucalyptus} \\ \frac{8!}{2! \cdot 5!} = 168 \end{array} \right\} \begin{array}{l} \text{\# with 3 eucalyptus not together} \\ 2520 - 168 = \boxed{2352} \text{ arrangements} \end{array}$$

$\checkmark$ calculates the number with all 3 adjacent
 $\checkmark$ applies complement to solve

- c) If the trees are to be planted in two rows (where the back row has 6 trees, and the front row has 4 trees), how many arrangements can be made if the three eucalyptus trees must all be in the front row?

(3 marks)

case (i) golden wattle at front (i.e. 5 paperbark & 1 golden wattle at back)

$$\frac{4!}{3!} \times \frac{6!}{5!} = 24$$

case (ii) paperbark at front

$$\frac{4!}{3!} \times \frac{6!}{2! \cdot 4!} = 60$$

$$\begin{aligned} \therefore \text{Total \#} &= 24 + 60 \\ &= \boxed{84} \text{ arrangements.} \end{aligned}$$

$\checkmark$ calculates for case 1
 $\checkmark$ calculates for case 2
 $\checkmark$ total
 F/T for total only if cases are appropriate